Research Paper | Mathematics | E-ISSN No : 2454-9916 | Volume : 9 | Issue : 9 | September 2023

FUNCTIONAL IDENTITIES ON PRIME RINGS INVOLVING GENERALIZED DERIVATIONS

Dr. Krishna Gopal Pradhan

Assistant Professor, Department of Mathematics, Ramnagar College, Depal, Purba Medinipur, West Bengal, INDIA

ABSTRACT

The commutativity of a prime ring and structure maps defined on it is studied here. In particular, we prove that a prime ring R is either commutative or a nonzero generalized derivation F defined on it is the identity map whenever it satisfies the certain functional identities on R.

KEYWORDS: Prime Rings, Generalized Derivations, Functional Identities With Central Value.

INTRODUCTION

Through-out this article R will denote a ring with the center $Z(R)$. Let us now define some terminologies first. The ring R is said to be a *prime ring* if for any two elements $a, b \in R$, $aRb = 0$ implies either $a = 0$ or $b = 0$. The ring R is said to be a *semiprime ring* if for any element $a \in R$, $aRa = 0$ implies $a = 0$. From the definition, it follows that every prime ring is semiprime ring while converse is not necessarily true. Let $d: R \to R$ be a map on R. The map d is called an *additive map* if $d(x + y) = d(x) + d(y), \forall\, x, y \in R$. The map d is said to be a *derivation* on R if it is an additive and $d(xy) = d(x)y + xd(y), \forall\, x, y \in R$. A function $F: R \to R$ is called a *generalized derivation* of R if F is an additive map and there is a derivation d of R such that $F(xy) = F(x)y + xd(y), \forall\, x, y \in R$. In this case we say F is a generalized derivation of R associated with the derivation d. It can be observed that any derivation is a generalized derivation but converse is not true. Therefore, any property that is hold by a generalized derivation also hold by a derivation.

For examples of derivation and generalized derivation, let us consider a ring $R = \left\{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \mid a, b \in \mathbb{Z}\right\}$ and the mappings $d\,\&\,F: R \to R$ defined by $d\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$ & $F\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & a+b \\ 0 & 0 \end{bmatrix}$ for all $a, b \in \mathbb{Z}$. Then F is a generalized derivation of R associated with the derivation d.

Let S be a non empty subset of a ring R. Then a mapping $F: R \to R$ is said to *act as homomorphism* on S if $F(xy) = F(x)F(y), \forall\, x, y \in S$. Again, the map F is said to act as *anti-homomorphism* on S if $F(xy) = F(y)F(x), \forall\, x, y \in S$. The subsets $r(S) := \{x \in R: sx = 0, \forall\, s \in S\}$ and $l(S) := \{x \in R: xs = 0, \forall\, s \in S\}$ of R are called *right annihilator* of S and *left annihilator* of S respectively. The set $ann_R(S) := \{x \in R: xs = 0 = sx, \forall\, s \in S\}$ is called annihilator of S. Observe that $ann_R(S)$ is an ideal of R. For any $x, y \in R$, *commutator* or *Lie product* of $x\,\&\,y$, denoted by $[x, y]$, is defined by $[x, y] = xy - yx$. For any $x, y \in R$, *anticommutator* or *Jordan product* of $x\,\&\,y$, denoted by $x \circ y$, is defined by $x \circ y = xy + yx$. The following identities can be easily verified:

$$i)\ [xy, z] = x[y, z] + [x, z]y$$

$$ii)\ [x, yz] = y[x, z] + [x, y]z$$

The study of derivation got its momentum soon after the beautiful works by Posner E.C and Herstein I.N in their respective articles published in 1957. Posner [13] investigated a functional identity involving a derivation satisfied on a prime ring and got a nice result showing the connection between structure of the ring and structure of mapping defined on it. He proved that if a prime ring R has a nonzero derivation d such that $[d(x), x] \in Z(R), \forall x \in R$, then the prime ring R is commutative. Since then, many researchers have been studying the functional identities in several directions generalizing the above result. Interested readers may refer to [2], [3], [5],[6],[8],[10],[11],[12],[15].

Previous work:

In this article few results concerning commutativity of a

suitable subset of a prime ring and structure of the generalized derivation which satisfies some functional identity will be studied. To understand the generalization of the work, let us look at the development of the literature in this line of work.

Bell and Kappe [4] investigated a derivation of a prime ring that act as homomorphism or anti-homomorphism on a nonzero right ideal and got that the derivation is the zero map. After that in 2004, Rehman N. [14] generalized the above result by taking generalized derivation instead of derivation. He proved that if a generalized derivation F of a 2-torsion free prime ring R associated with a derivation d act as homomorphism or anti-homomorphism on a nonzero ideal I of R then either $d = 0$ or R is commutative. The result of Rehman was extended by Dhara B. [7] by taking R as semiprime ring. Later on, Albas E. [1] improved the result of Rehman by taking the functional identities $F(xy) \pm F(x)F(y) \in Z(R)$ and $F(xy) \pm F(y)F(x) \in Z(R)$ for all $x, y \in I$. Albas proved the following two theorems:

Theorem A: Let R be a prime ring with center $Z(R)$. If R has a nonzero generalized derivation F with associated derivation d such that

$$F(xy) \pm F(x)F(y) \in Z(R)\, \forall\, x, y \in I$$

where I is a nonzero ideal of R, then either R is commutative or $F = \pm I_{id}$ – the identitiy map on R.

In 2013, Dhara et al [9] proved the following theorem:

Let R be prime ring, J be a nonzero ideal of R and F a nonzero left centralizer map of R. (i) If $F(xy) - F(y)F(x) \in Z(R)\, \forall\, x, y \in J$, then either R is commutative or F is identity map. (ii) If $F(xy) + F(y)F(x) \in Z(R)\, \forall\, x, y \in J$, then either R is commutative or $F(r) = -r, \forall\, r \in R$.

In this article, we have studied about the structure of the prime ring R and generalized derivation defined on R satisfying the identity

$$F(xy) \pm F(x)F(y) + z \in Z(R)\, \forall\, x, y, z \in R$$

Main Results:

In this section, we prove our main results.

Theorem 1: Let R be a prime ring and F be a nonzero generalized derivation of R associated with a derivation d of R satisfying

$$\boldsymbol{F(uv) + F(u)F(v) + w \in Z(R), \forall\, u, v, w \in R.}$$

Then either R is commutative or $F = -I_{id}$, the identity map on R.

Proof: By hypothesis we have,

$$F(uv) + F(u)F(v) + w \in Z(R), \forall\, u, v, w \in R.$$

Commuting with w, we get

$$[F(uv) + F(u)F(v), w] = 0, \forall\, u, v, w \in R. \qquad (*)$$

Let us put $v = vw$ in the above equation and we get

$$[F(uvw) + F(u)F(vw), w] = 0, \forall\, u, v, w \in R.$$

Using the definition of generalized derivation, we have

$$[F(uv)w + uvd(w) + F(u)\{F(v)w + vd(w)\}, w] = 0, \forall\, u, v, w \in R.$$

Which implies

$$\left[\{F(uv) + F(u)F(v)\}w + \left(u + F(u)\right)vd(w), w\right] = 0, \forall\, u, v, w \in R.$$

$$\Rightarrow [\{F(uv) + F(u)F(v)\}w, w] + \left[\left(u + F(u)\right)vd(w), w\right] = 0, \forall\, u, v, w \in R.$$

$$\Rightarrow \left[F(uv) + F(u)F(v), w\right]w + \left[\left(u + F(u)\right)vd(w), w\right] = 0, \forall\, u, v, w \in R.$$

Using (*), we get

$$\left[\left(u + F(u)\right)vd(w), w\right] = 0, \forall\, u, v, w \in R. \qquad (1)$$

Putting here $u = uw$ and simplifying, we get

$$\left[\left(u + F(u)\right)wvd(w), w\right] + [ud(w)vd(w), w] = 0, \forall\, u, v, w \in R. \qquad (2)$$

Putting $v = wv$ in (1), we get

$$\left[\left(u + F(u)\right)wvd(w), w\right] = 0, \forall\, u, v, w \in R.$$

Using this from (2) we get

$$[ud(w)vd(w), w] = 0, \forall\, u, v, w \in R. \qquad (3)$$

After putting $u = d(w)u$ in (3) we see that

$$[d(w)ud(w)vd(w), w] = 0, \forall\, u, v, w \in R.$$

Now,

$$0 = [d(w)ud(w)vd(w), w]$$

$$= d(w)[ud(w)vd(w), w] + [d(w), w]ud(w)vd(w).$$

Using (3) here, we get

$[d(w), w]ud(w)vd(w) = 0, \forall\, u, v, w \in R.$ (4)

Since R is a prime ring, from (4) we get either $d(w) = 0, \forall\, w \in R$ or $[d(w), w] = 0, \forall\, w \in R$. That is either $d = 0$ or R is a commutative ring (the later conclusion arises from the celebrated Posner's theorem [13]).

If $d = 0$, then $F(xy) = F(x)y, \forall\, x, y \in R$, that is F is left multiplier map of R. In this case, the hypothesis reduces to

$F(u)v + F(u)F(v) + w \in Z(R), \forall\, u, v, w \in R.$

Commuting with w, we get

$0 = [F(u)v + F(u)F(v), w] = [F(u)(v + F(v)), w], \forall\, u, v, w \in R.$ (5)

Replacing $v = vz$, where $z \in R$, we get

$0 = [F(u)(vz + F(vz)), w], \forall\, u, v, w, z \in R.$

$0 = [F(u)(v + F(v))z, w] = [F(u)(v + F(v)), w]z + F(u)(v + F(v))[z, w]$

$= F(u)\big(v + F(v)\big)[z, w], \forall\, u, v, w, z \in R.$ (Using (5))

Putting $u = ur, r \in R$ and using the fact that F is left multiplier, we get

$F(u)r\big(v + F(v)\big)[z, w] = 0, \forall\, u, v, w, z, r \in R$. Since R is a prime ring and F is nonzero, $\big(v + F(v)\big)[z, w] = 0, \forall\, v, w, z \in R$. Again replacing v by $vr, r \in R$ we get, $\big(v + F(v)\big)r[z, w] = 0, \forall\, v, w, z, r \in R$. Using the definition for prime ring, we get either $[z, w] = 0, \forall\, w, z \in R$ or $\big(v + F(v)\big) = 0, \forall\, v \in R$. That is either R is commutative or $F(v) = -v, \forall\, v \in R$, in other words $F = -I_{id}$, where I_{id} is the identity map on R.

Theorem 2: Let R be a prime ring and F be a nonzero generalized derivation of R associated with a derivation d of R satisfying

$$\boldsymbol{F(uv) - F(u)F(v) + w \in Z(R), \forall\, u, v, w \in R.}$$

Then either R is commutative or $F = I_{id}$, the identity map on R.

Proof: Let us consider the map $-F, -d: R \to R$ defined by

$(-F)(x) = -F(x)$ & $(-d)(x) = -d(x), \forall\, x \in R$. Now $(-d)(xy) = -d(xy) = -d(x)y - xd(y)$ and

$(-F)(xy) = -F(xy) = -F(x)y - xd(y) = (-F)(x)y + x(-d)(y), \forall\, x, y \in R.$

These imply that $-F$ is a generalized derivation on R associated with $-d$.

By given hypothesis,

$-(-F)(uv) - (-F)(u)(-F)(v) - w \in Z(R), \forall\, u, v, w \in R$

$\Rightarrow (-F)(uv) + (-F)(u)(-F)(v) + w \in Z(R), \forall\, u, v \in R$

as $Z(R)$ is a subring. So, by previous theorem either, R is commutative or $-F = -I_{id}$, that is, $F = I_{id}$, where I_{id} is the identity map on R. Hence proved.

REFERENCES

1. E. Albas, "Generalized derivations on ideals of prime rings", Miskolc Mathematical Notes, vol. 14, no. 1, pp. 3–9, 2013.
2. A. Ali, N. Rehman, and S. Ali, "On lie ideals with derivations as homomorphisms and anti-homomorphisms", Acta Math. Hungar, vol. 101, no. 1-2, pp. 79–82, 2003.
3. S. Ali and S. Huang, "On generalized Jordan (α,β)-derivations that act as homomorphisms or antihomomorphisms," Journal of Algebra and Computational Applications, vol. 1, no. 1, pp. 13–19, 2011.
4. H. E. Bell and L. C. Kappe, "Rings in which derivations satisfy certain algebraic conditions," Acta Math. Hungar, vol. 53, pp. 339–346, 1989.
5. H. E. Bell and W. S. Martindale-III, "Centralizing mappings of semiprime rings," Canad. Math. Bull., vol. 30, no. 1, pp. 92–101, 1987.
6. M. N. Daif and H. E. Bell, "Remarks on derivations on semiprime rings," Internat. J. Math. Math. Sci., vol. 15, no. 1, pp. 205–206, 1992.
7. B. Dhara, "Generalized derivations acting as a homomorphism or anti-homomorphism in semiprime rings," Beitr. Algebra Geom., vol. 53, pp. 203–209, 2012.
8. B. Dhara and S. Ali, "On n-centralizing generalized derivations in semiprime rings with applications to C^*-algebras," J. Algebra and its Applications, vol. 11, no. 6, pp. Paper No. 1 250 111, 11 pages, 2012.
9. B. Dhara, A. Fosner, and S. Mondal, "Left multipliers on lie ideals in prime rings," Journal of Advanced Research in Pure Mathematics, vol. 5, no. 4, pp. 57–64, 2013.
10. V. D. Filippis, "Generalized derivations as Jordan homomorphisms on lie ideals and right ideals," Acta Math. Sinica, English Series, vol. 25, no. 2, pp. 1965–1974, 2009.
11. I. Gusic, "A note on generalized derivations of prime rings," Glasnik Mate., vol. 40, no. 60, pp. 47–49, 2005.
12. I. N. Herstein, Rings with involution. Chicago: Univ. of Chicago Press, 1976.
13. E. C. Posner, "Derivations in prime rings", Proc. Amer. Math. Soc. Vol. 8, pp.1093-1100, 1957.
14. N. Rehman, "On generalized derivations as homomorphisms and anti-homomorphisms," Glasnik Mate., vol. 39, no. 59, pp. 27–30, 2004.
15. Y. Wang and H. You, "Derivations as homomorphisms or anti-homomorphisms on lie ideals," Acta Math. Sinica, English Series, vol. 23, no. 6, pp. 1149–1152, 2007.